An example

Let C be the genus-2 curve over $\mathbb{Q}$ defined by

$$y^2 + (2x^3 - 3x^2 - 41x + 110)y = x^3 - 51x^2 + 425x + 179.$$

Let D be the divisor $(2, 17) + (4, 23) - \infty_1 - \infty_2$.

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Record-breaking result:

The class of D in Jac C is a torsion point of order 70.

Split Jacobians

Idea:

Construct a curve whose Jacobian is $(3, 3)$-isogenous to $E_1 \times E_2$, where

- E_1 has a torsion point of order 7, and
- E_2 has a torsion point of order 10.

Constructing curves $(3, 3)$-isogenous to $E_1 \times E_2$:

Formulas essentially due to Goursat (1885).
More complicated formulas by Kuhn (1988) and Shaska (2004).
Proof that Goursat's formulas have nice arithmetic properties: Bröker–H.–Lauter–Stevenhagen (2014).

Main requirement: Need $E_1[3] \cong E_2[3]$ as Galois modules.

Finding an example

Need E_1, E_2 such that

- E_1 has a torsion point of order 7,
- E_2 has a torsion point of order 10, and
- $E_1[3] \cong E_2[3]$.

Searched Cremona tables, Stein–Watkins tables, parametrized families.

Found one example: $E_1 = 858\text{k}1$ and $E_2 = 66\text{c}2$.