Are there ∞ly many polynomials $P : \mathbf{Z} \to \mathbf{Z}$ such that $|P(n)| \leq \text{Fib}_n$ for each $n \geq 0$?

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Let $A \subset \mathbf{Q}[X]$ be the ring

$$\left\{ \sum_{i=0}^{d} a_i \binom{X}{i} : d < \infty,\ a_i \in \mathbf{Z} \right\}$$

of integer-valued polynomials.

Say that a function $f : \{0, 1, 2, \ldots\} \to \mathbf{R}$ "accommodates A" if $\exists$ infinitely many $P \in A$ s.t. $|P(n)| \leq f(n)$ for each $n = 0, 1, 2, \ldots$.

Example: 2^n accommodates A because each $\binom{X}{i}$ satisfies $0 \leq \binom{n}{i} \leq 2^n$ for all $n \geq 0$.

Question:

For which $C > 1$, $r > 0$ does $f(n) = rC^n$ accommodate A?

Question (repeat):

For which $C > 1$, $r > 0$ *does* $f(n) = rC^n$ *accommodate* A?

Nearly-complete answer: all r if $C > \varphi$, and none if $C < \varphi$, where $\phi =$ Golden Ratio $(1 + \sqrt{5})/2 = 1.61803\ldots$.

Proof: See Mathoverflow question 139140. Hint: start with the generating function $\sum_{n=0}^{\infty} P(n)z^n \in (1-z)^{-1}\mathbf{Z}[(1-z)^{-1}]$.

Question: what happens for $C = \varphi$ itself? In particular, does $f(n) = \mathrm{Fib}_n$ (Fibonacci) accommodate A?

Numerical evidence: for $r = 1$ the number of P's seems to blow up fast as $C \to \varphi$ from below: with $a_d > 0$, only three for $C = \sqrt{2}$ (namely 1, $X - 1$, and $\frac{X^2-5X+2}{2} = \binom{X}{2} - 2\binom{X}{1} + \binom{X}{0}$); ten for $C = 3/2$, with degree as large as 4 for $\binom{X}{4} - 3\binom{X}{3} + 3\binom{X}{2} - \binom{X}{1}$; and then (thanks to `forqfvec`) ...

C	1.52	1.53	1.54	1.55	$14/9 = 1.55555\ldots$	$25/16 = 1.5625$
#	14	25	37	57	89	144

with degrees as high as 12, e.g. setting $a_0 = a_1 = a_2 = 0$ and

$$(a_3, a_4, a_5, \ldots, a_{12}) = (1, -8, 29, -63, 91, -91, 63, -29, 8, -1)$$

(coefficients of $(u-1)^7(u^2-u+1)$) yields $\sup_{n \geq 0} |P(n)|^{1/n} = 1.55549938\ldots$ at $n = 36$. So at least the Lucas numbers should accommodate A?! $\diamondsuit$